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Abstract

In a general class of discrete optimization problems, some of the elements may have random costs associated with them. In such a situation, the notion of optimality needs to be suitably modified. In this work we define an optimal solution to be a feasible solution with the minimum risk. We focus on the min-sum objective function, for which we prove that knowledge of the mean values of these random costs is enough to reduce the problem into one with fixed costs. We discuss the implications of using sample means when the true means of the costs of the random elements are not known, and explore the relation between our results and those from post-optimality analysis. We also show that discrete optimization problems with min-max objective functions depend more intricately on the distributions of the random costs.

Keywords: min-sum, min-max, risk, optimality, postoptimality analysis

AMS Subject Classification 90C27, 90C15

1. Introduction

In discrete optimization problems (DOPs), it is more of a norm than an exception that the costs of some of the problem elements are not fixed. The practical solution in most of such cases is to assume some "good" approximation of the data and solve the problem. Once an optimal solution is obtained, post-optimality analysis techniques like sensitivity analysis are typically used to gain insight into the robustness of the solution obtained. In many situations, however, the decision maker has a fairly good idea about the randomness of these elements. In this work, we try to find out how information about the probability distribution of the costs of the random elements can be used to aid decision making for DOPs. In general, such problems can be considered to be integer programming problems with stochastic coefficients, which are known to be more difficult computationally than their fixed cost counterparts.

We consider a DOP Π as a collection of problem instances $\pi = (G, \mathbb{S}, z)$, where G is a finite ground set, with each element $e \in G$ having an associated cost c_e. The set $\mathbb{S}$, ($\subseteq 2^{|G|}$) of feasible solutions, is usually not described explicitly, but rather by a set of rules that each $S \in \mathbb{S}$ must satisfy. The function $z : \mathbb{S} \rightarrow \mathfrak{R}$ is referred to as the objective function (or the cost function), and the optimization problem is one of finding a member of $\arg\min_{S\in\mathbb{S}}\{z(S)\}$. In this paper, we primarily limit ourselves with *min-sum* objective functions, i.e. cases where $z(S) = \sum_{e\in S} c_e$. Such a generic framework covers a wide variety of discrete optimization problems as shown in the following examples.

Example 1 (Minimum Spanning Tree Problem) *Consider an undirected graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V}$ is the set of vertices and $\mathbb{E}$ is the set of edges, with each edge $e \in \mathbb{E}$ having a length associated with it. The goal of this problem is to find a spanning tree (a tree that connects all $v \in \mathbb{V}$) in the graph with minimum possible combined length. This fits into our general formulation above, with an edge in the graph referring to an element (so that $\mathbb{E}$ corresponds to the set G). c_e is the length of edge e, $\mathbb{S}$ is the set of all spanning trees in the graph, and $z(S) = \sum_{e\in S} c_e$ represents the total length of the edges included in the spanning tree S.* □

Example 2 ($0/1$ Knapsack Problem) *We are given a set of r elements $\mathbb{E} = \{e_1, \ldots, e_r\}$, each element e_j having an associated profit p_j and an associated weight w_j; and a capacity B. We are to determine a subset of elements in $\mathbb{E}$ with maximum combined profit whose combined weight does not exceed the capacity. In our notation, G refers to the set $\mathbb{E}$, $c_{e_j} = p_j$ for all $e \in \mathbb{E}$, $\mathbb{S}$ is the set of all $S \subseteq \mathbb{E}$ such that $\sum_{e_j\in S} w_j \leq B$, and $z(S) = \sum_{e_j\in S} p_j$ for all $S \in \mathbb{S}$.* □

Example 3 (Symmetric Traveling Salesperson Problem) *In an undirected graph* $\mathbb{G} = (\mathbb{V}, \mathbb{E})$*, where each edge* $e \in \mathbb{E}$ *has a cost associated with it, we want to find a minimum cost Hamiltonian cycle. In our notation,* G *is the set* $\mathbb{E}$*, so that each edge in the graph is an element,* c_e *is the cost of edge* e *for all* $e \in \mathbb{E}$*,* $\mathbb{S}$ *is the set of all Hamiltonian cycles in* $\mathbb{G}$*, and* $z(S) = \sum_{e \in S} c_e$ *is the sum of the lengths of the edges in* S. □

We deal with the situation where the costs associated with certain elements are random variables. Therefore we need to suitably extend the standard notions of optimality. We will first formalize our problem through the following definitions and set up.

Since the costs of some of the elements in the problems we consider are not constant, we will classify the problem elements using the following notation:

Definition 1 *An element* $e \in G$ *in* $\pi = (G, \mathbb{S}, z)$ *is called* fixed *(alternatively* random*) if* c_e *is constant (alternatively random valued).*

Definition 2 *Given any fixed set of values for* c_e*'s, the* loss *associated with a solution* $S \in \mathbb{S}$ *is defined by*

$$L(S) = z(S) - Z^*,$$

where Z^* *is the minimum possible value of the objective function for given values of* c_e*'s (and hence is a function of these* c_e*'s).*

Obviously, with some of the c_e's being random, the loss of any feasible solution S is also a random variable. In practice, it would not be desirable to adopt a new course of action with every alteration of the c_e's, especially if we deal with $\mathcal{NP}$-hard problems. So, we need to find a solution which would be "good" regardless of the realization of the costs of the random elements. With this in mind, we define the risk associated with a solution in the following manner:

Definition 3 *The* risk *associated with a solution* $S \in \mathbb{S}$ *is given by*

$$R(S) = \mathbb{E}L(S) = \mathbb{E}[z(S) - Z^*],$$

where the expectation is taken with respect to the costs of the random elements.

We define the optimization problem for DOPs with random elements as the problem of finding a feasible solution with minimum risk. Notice that if all the elements of the instance are fixed, the minimum risk solution corresponds to the traditional concept of an optimal solution, i.e., a least cost solution.

In the next section we analyze DOPs with min-sum objective functions. We show that knowledge of the mean of the distribution functions of the costs of the random elements is sufficient to obtain an optimal solution for these problems. Section 3 is divided into three subsections. In Subsection 3.1 we consider, mainly, the implication of using sample means instead of true means of the random elements in DOPs with min-sum objective functions. Then in Subsection 3.2 we discuss the connection between our results and those from post-optimality analysis. Finally, in Subsection 3.3 we show that DOPs with min-max objective functions depend more intricately on the distribution functions of any random cost elements they contain. We conclude this paper with Section 4 containing a summary of our results and possible directions for future research.

2. DOPs with Min-Sum Objective

In this section we analyze DOPs with min-sum objective functions and having one or more random elements. We first consider DOPs with one random element and show that knowledge of the mean of the distribution function for this element's cost is sufficient to obtain an optimal solution. We then generalize the result for DOPs with an arbitrary number of random elements.

2.1 DOPs with one random element

Let $\pi = (G, \mathbb{S}, z)$ be a DOP instance with a single random element $e \in G$. First, we study the least cost objective function value (Z^*) as a function of c_e.

Let us denote the cost of the random element e by a random variable X. Let X have a cumulative distribution function $H(.)$ with mean μ, i.e. $H(x) = P(X \leq x)$, and $\mu = \int x dH(x)$. We split the set of all feasible solutions $\mathbb{S}$ into $\mathbb{S}_e$ and $\mathbb{S}^e$, respectively consisting of all solutions containing e, and of all solutions not containing e. Let S_e be a least cost solution in $\mathbb{S}_e$ and S^e be a least cost solution in $\mathbb{S}^e$. We note that while S_e and S^e need not be unique, they remain least cost solutions in their respective groups regardless of the value of c_e. This is because, a change in c_e does not affect the cost of any solution in $\mathbb{S}^e$, while it affects all solutions in $\mathbb{S}_e$ by the same amount.

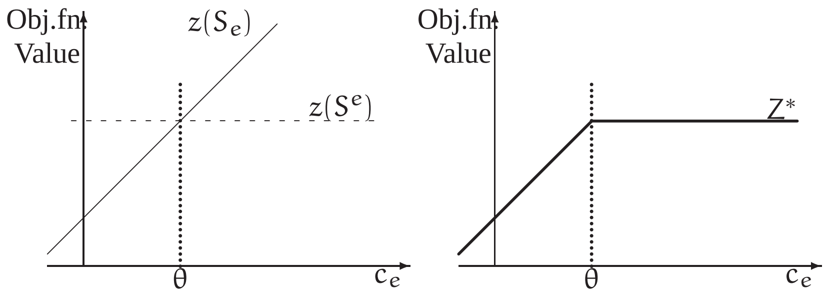

Figure 1: $z(S_e)$, $z(S^e)$, and Z^* as a function of c_e (min-sum objective)

For extreme possible low values of c_e, typically, $z(S_e) < z(S^e)$. (Otherwise, the randomness or otherwise of c_e is not an issue at all, since e would not be included in the optimal solution in any case.) When c_e increases, the cost of all solutions in $\mathbb{S}_e$ increase while the cost of all solution in $\mathbb{S}^e$ remain the same. So S_e remains optimal until c_e increases to become larger than some threshold value, say θ, when $z(S_e)$ becomes equal to $z(S^e)$. If c_e increases further, $z(S_e) > z(S^e)$, and S^e becomes a new optimal solution. Clearly, no further increase in c_e will make S^e suboptimal. We see therefore, that $Z^*(c_e)$ is a continuous function with a slope of 1 when $c_e < \theta$ and a slope of 0 when $c_e > \theta$ (see Figure 1).

In the above, while the threshold value, θ, of the random cost is introduced in terms of a specific choice of the least cost solutions (which need not be unique) within the two groups, its properties remain more general. In other words, a least cost solution contains e if and only if $c_e \leq \theta$ (and at $c_e = \theta$, both S_e and S^e are optimal in the least cost sense).

Then, for any value of c_e, at least one of S_e and S^e is a least cost solution. Accordingly, in adopting S_e as a solution, one incurs a loss equal to $(x-\theta)$ if $c_e(= x) > \theta$. Similarly, by taking S_e as a solution, there is a loss of $(\theta - x)$ when $x < \theta$. Thus, the risk of these two solutions are

$$R(S_e) = \int_{\theta}^{\infty} (x - \theta) dH(x); \quad R(S^e) = \int_{-\infty}^{\theta} (\theta - x) dH(x). \tag{1}$$

From (1), we observe that,

$$R(S_e) \leq R(S^e) \iff \int_{-\infty}^{\infty} (x-\theta)dH(x) \leq 0 \iff \mu \leq \theta,$$

where μ is the mean of the probability distribution of c_e. The argument above shows:

Theorem 1 *If c_e has a finite mean μ, then a feasible solution, which is a least cost solution when $c_e = \mu$, has the least possible risk.*

REMARK. It is easy to see that if X has a finite support on $[a, b]$, then

$$\begin{aligned} R(S_e) &= (b-\theta) - \int_{\theta}^{b} H(x)dx \\ R(S^e) &= \int_{a}^{\theta} H(x)dx \end{aligned}$$

Thus,

$$R(S_e) \leq R(S^e) \iff (b-\theta) \leq \int_{a}^{b} H(x)dx, \tag{2}$$

and the right hand side of the inequality in (2) reduces to $b - \mu$, using integration by parts. This presents an alternative proof of the Theorem 1 for this special case.

REMARK. It is easy to see that (1) reduces to

$$R(S_e) = \int_{\theta}^{\infty} (1 - H(t))dt; \quad R(S^e) = \int_{-\infty}^{\theta} H(t)dt.$$

In this respect, they look similar to expected shortage and surplus functions encountered in the stochastic programming literature (refer for example, to Klein Haneveld and Van der Vlerk [3], p. 48–49). Note that in our analysis, θ is not a decision variable but is determined by the data in a given instance.

Theorem 1 implies that for DOPs with one random element, a knowledge of the mean of the distribution function of the cost of the random element is enough to compute an optimal solution. Let us next generalize this result to DOPs with more than one random elements.

2.2 DOPs with more than one random elements

We now consider a DOP instance π with an arbitrary but fixed number k (> 1) of random elements. Accordingly, we partition G into $G_R = \{e_1, \ldots, e_k\}$ of random elements, and $G_F = \{e_{k+1}, \ldots, e_n\}$ of fixed elements. Let $X_1, \ldots, X_k$ be the random variables denoting the values of $c_{e_1}, \ldots, c_{e_k}$ and $H(x_1, \ldots, x_k)$ denote $\Pr(X_1 \leq x_1, \ldots, X_k \leq x_k)$. We represent the objective function value of any solution S as

$$z(S) = F(S) + \sum_{i:e_i \in S \cap G_R} X_i \tag{3}$$

where $F(S) = \sum_{e \in S \cap G_F} c_e$ is the fixed component of the cost $z(S)$.

Let $K_1, \ldots, K_{2^k}$ be the 2^k subsets of $K = \{1, \ldots, k\}$. For $i = 1, \ldots, 2^k$, let

$$\mathbb{S}_i = \{S : S \in \mathbb{S}; \quad e_j \in S \quad \forall j \in K_i; \quad e_j \notin S \quad \forall j \in K \setminus K_i, \} \tag{4}$$

constitute a partition of $\mathbb{S}$. In certain problem situations, some of the $\mathbb{S}_i$'s may be empty.

Lemma 1 *If $S^1, S^2 \in \mathbb{S}_i$, for some i, then $z(S^1) - z(S^2)$ is non-random.*

PROOF. By construction (4), S^1 and S^2 have the same set of random elements and hence by (3) $z(S^1) - z(S^2) = F(S^1) - F(S^2)$ which is non-random. □

For any fixed set of costs $(x_1, \ldots, x_k)$, let S_i denote a least cost solution within $\mathbb{S}_i$. While S_i need not be unique, by Lemma 1, it remains a least cost solution among the ones in $\mathbb{S}_i$ regardless of values of the cost variables X_i's.

The following lemma is useful for restricting our search for optimal solutions.

Lemma 2 *For any solution* $S \in \mathbb{S}$, $R(S) \geq \min_j\{R(S_j)\}$.

PROOF. Since $\mathbb{S} = \bigcup_{i=1}^{2^k} \mathbb{S}_i$, $\exists j$ such that $S \in \mathbb{S}_j$. Then

$$R(S) = \mathbb{E}[z(S) - Z^*] = \mathbb{E}[z(S) - z(S_j) + z(S_j) - Z^*] = z(S) - z(S_j) + R(S_j) \geq R(S_j),$$

following Lemma 1 and the selection of S_j. □

An immediate implication of Lemma 2 is the fact that at least one among S_1 through S_{2^k} is an optimal solution in the minimum risk sense.

Let us introduce the sets $\{\mathcal{R}_i; 1 \leq i \leq 2^k\}$ in the k-dimensional Euclidean space ($\Re^k$) through

$$\mathcal{R}_i = \{(x_1, \ldots x_k) : S_i \text{ is a least cost solution at } (x_1, \ldots x_k)\}, \tag{5}$$

and a partition through $\{P_i; 1 \leq i \leq 2^k\}$ where

$$P_1 = \mathcal{R}_1, \quad \text{and } P_i = \mathcal{R}_i \setminus (\bigcup_{j<i} P_j) \quad i = 2, \ldots, 2^k. \tag{6}$$

Notice that for all $i = 1, \ldots, 2^k$,

$$P_i \subseteq \mathcal{R}_i. \tag{7}$$

Now by (3),

$$\begin{aligned} z(S_i) - z(S_j) &= F(S_i) + \sum_{m \in K_i} x_m - \Big[F(S_j) + \sum_{m \in K_j} x_m\Big] \\ &= \Big[\sum_{m \in K_i \setminus K_j} x_m - \sum_{m \in K_j \setminus K_i} x_m\Big] + F(S_i) - F(S_j). \end{aligned} \tag{8}$$

If S_i is a least cost solution at $(x_1, \ldots x_k)$, then for this set of costs, $z(S_i) \leq z(S_j), \quad \forall j = 1, \ldots, 2^k$. So, an alternative characterization of $\mathcal{R}_i$ is

$$\mathcal{R}_i = \Big\{(x_1, \ldots, x_k) : \sum_{m \in K_i \setminus K_j} x_m - \sum_{m \in K_j \setminus K_i} x_m \leq F(S_j) - F(S_i), \quad j = 1, \ldots, 2^k\Big\} \tag{9}$$

We are now in a position to prove the main theorem for DOPs with more than one random elements.

Theorem 2 *If $X_1, \ldots, X_k$ are random variables having finite means $\mu_1, \ldots, \mu_k$ respectively, then the least cost tour, corresponding to the costs of $c_{e_1}, \ldots, c_{e_k}$ fixed at $\mu_1, \ldots, \mu_k$, will be optimal in the least risk sense.*

PROOF. Since $\{P_j\}$ form a partition of $\mathfrak{R}^k$ and $S_i \in P_i$ is a least cost solution in P_i, the risk associated with S_i can be written as

$$R(S_i) = \sum_{j=1}^{2^k} \int_{P_j} \{z(S_i) - z(S_j)\} dH(x_1, \ldots, x_k) \tag{10}$$

$$\begin{aligned}
\text{So, } & R(S_i) - R(S_j) \\
= & \sum_{m \neq i,j} \int_{P_m} \{z(S_i) - z(S_j)\} dH(x_1, \ldots, x_k) \\
& + \int_{P_j} \{z(S_i) - z(S_j)\} dH(x_1, \ldots, x_k) \\
& - \int_{P_i} \{z(S_j) - z(S_i)\} dH(x_1, \ldots, x_k) \\
= & \int_{\mathfrak{R}^k} \big\{z(S_i) - z(S_j)\big\} dH(x_1, \ldots, x_k) \\
= & \int_{\mathfrak{R}^k} \Big[\sum_{m \in K_i \setminus K_j} x_m - \sum_{m \in K_j \setminus K_i} x_m - (F(S_j) - F(S_i)) \Big] dH(x_1, \ldots, x_k), \text{ by (8)} \\
= & \sum_{m \in K_i \setminus K_j} \mu_m - \sum_{m \in K_j \setminus K_i} \mu_m - (F(S_j) - F(S_i)). \qquad (11)
\end{aligned}$$

Hence, for any i,

$$R(S_i) = \min_{1 \le j \le 2^k} R(S_j) \iff R(S_i) \le R(S_j) \quad \forall j \iff (\mu_1, \ldots, \mu_k) \in \mathcal{R}_i,$$

by (9) and (11). □

Theorem 2 tells us that knowledge of the means of the costs of the random elements is adequate to obtain an optimal tour for a generic DOP with a min-sum objective function.

3. Discussions

In Section 2 we have shown that we can find "good" solutions to DOPs with min-sum objectives having random elements with just the knowledge of the mean values of the random costs. Here, we first show in Subsection 3.1 that we can obtain a reasonably good solution even if we have the mean values of a few past realizations of the problem instance. We then examine the connections of our results in Section 2 with those in post-optimality analysis in Subsection 3.2. We close the section with Subsection 3.3 in which we show that DOPs with min-max objectives depend much more intricately on the distribution functions of the costs of their random elements than their counterparts with min-sum objectives.

3.1 Statistical Perspectives

Theorems 1 and 2 are significant because they tell us that even if one has the knowledge about the only the means of the costs of the random elements, and nothing else about the randomness, one can find the best solution in the minimum risk sense at least for the min-sum objective function. However, it is also possible that in some cases, one would not have any (or very reliable) information about the true means. Instead, the data regarding the actual costs for the past few times (i.e. recent realizations of these random variables or random observations) would be available. The natural choice then would be to use the sample mean instead of μ in determining the optimal solution. Since, sample mean is random, one cannot guarantee that a least risk solution would be obtained in such a case. In this subsection, we show that in spite of that, the situation is very ideal in asymptotic sense.

We work under the assumption that we have n independent and identically distributed observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$, on $\mathbf{X} = (X_1, \ldots, X_k)$, where each $\mathbf{X}_i$ is k-variate. We propose to use

$$\bar{\mathbf{X}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i, = (\bar{X}_1, \ldots, \bar{X}_k)$$

instead of μ, when the latter is unknown.

Let S_0 be a minimum risk solution and $Z_0 = z(S_0)$ be its objective value. By Theorem 2, $z(S_0) = z(S_i)$ for one of the S_i's as defined in Subsection 2.2. Let us assume, without loss of any generality that, $\mu \in P_{i_0}$, so that

$$z(S_0) = z(S_{i_0}) = F(S_{i_0}) + \sum_{m \in G_R \cap K_{i_0}} X_m.$$

By the Kolmogorov's Strong Law of Large Numbers (SLLN), the sample mean is consistent for population mean, i.e.

Result 1 $\bar{\mathbf{X}}_n \to \mu$, *almost surely, as* $n \to \infty$.

Let $\hat{Z}_n$ be the objective function value of the least cost solution when one uses $\mathbf{X} = \bar{\mathbf{X}}_n$. The following theorem shows that the difference between $\hat{Z}_n$ and Z_0 converges to 0 almost surely, i.e. the penalty for not knowing the true mean diminishes when n becomes large.

Theorem 2 $\hat{Z}_n - Z_0 \to 0$, *almost surely, as* $n \to \infty$.

PROOF. For any fixed n, define

$$A_{i,n} = \{\bar{\mathbf{X}}_n \in P_i\}, \quad 1 \le i \le 2^k,$$

where P_i is as defined in (6). Since P_i's form a partition of $\mathfrak{R}^k$, for any given n, $\{A_{i,n} : 1 \le i \le 2^k\}$ form a partition of the probability space on which $\mathbf{X}_i$'s are defined. From the discussion in Subsection 2.2, it follows that if $\bar{\mathbf{X}}_n \in P_i$, then $\hat{Z}_n = z(S_i)$, where S_i's are as chosen in the proof of Theorem 2. Thus, we have,

$$\hat{Z}_n = \sum_{i=1}^{2^k} z(S_i)\mathbb{I}_{A_{i,n}}, \tag{12}$$

where $\mathbb{I}_A$ is the indicator (0-1) function of the set A.

$$\hat{Z}_n - Z_0 \;=\; \sum_{\substack{i=1\\ i\neq i_0}}^{2^k} \Big[z(S_i) - Z(S_{i_0})\Big]\mathbb{I}_{A_{i,n}}. \tag{13}$$

By SLLN (1), it follows

$$\mathbb{I}_{A_{i,n}} \overset{n\to\infty}{\longrightarrow} \begin{cases} 1 & \text{for } i = i_0 \\ 0 & \text{for } i \neq i_0 \end{cases} \quad \text{almost surely.} \tag{14}$$

And by (8), $z(S_i) - Z(S_{i_0})$ is a random variable, free of n, and hence stochastically bounded or $O_p(1)$. Combining (13) and (14), we get the desired result. □

REMARK. Since almost sure convergence for a sequence of indicator functions implies the convergence in probability to any order, by (14), we have

$$\mathbb{I}_{A_{i,n}} = o_p(n^{-\alpha}), \quad \text{for any } \alpha.$$

Thus, from (13), we can see that

$$\hat{Z}_n - Z_0 = o_p(n^{-\alpha}), \quad \text{for any } \alpha.$$

In other words, it is not possible to get any limiting *non-degenerate* distribution by multiplying $(\hat{Z}_n - Z_0)$ with any polynomial (indeed, also exponential, by the same logic) in n. The practical implication of this is that the loss in not knowing the true mean decreases to zero at a very fast rate in the asymptotic sense.

We conclude this subsection with the following statistical observations.

REMARK. From a Bayesian point of view, one may have knowledge about the prior probability distribution of $\mathbf{X}$ as well as data points depicting its realized values. In such a situation, one would obviously use the posterior mean $\mathbb{E}(\mathbf{X}|\bar{\mathbf{X}}_n)$ instead of μ or $\bar{\mathbf{X}}_n$.

REMARK. In statistical decision theory, 'loss' and 'risk' has the same relationship as introduced in this work. The choice of loss function (as in Definition 3) is very appealing and natural in the current context, but following statistical literature one may like to adopt other forms for this, and accordingly, the optimal solution would be different. For example, if we consider the square of current form of loss, then the characterisations in Theorems 1 and 2 do not remain valid.

3.2 Connections with Post-Optimality Analysis

DOPs with one random element have been widely studied in the literature on sensitivity analysis of DOPs. The tolerance approach to sensitivity analysis derives expressions for the interval within which the cost of the random element must lie for the current optimal solution to remain optimal. This interval is specified in terms of upper and lower tolerance limits, i.e. the maximum increase or decrease in the cost of the element that is allowable if the current solution is to remain optimal. Such studies have been carried out for generalized DOPs like the one described in the introductory section (refer, for example, to Ramaswamy & Chakravarti [5] and Van Hoesel & Wagelmans [10]). Greenberg [2] maintains an annotated bibliography on this literature. Even though sensitivity analysis is a post-optimality analysis technique and the result that we have described in this paper is an optimization result useful for finding an optimal solution, there are interesting connections between the two. Ramaswamy & Chakravarti [5] provide the following characterization of the tolerance limits for DOPs with min-sum objective functions.

Result 3 (Ramaswamy & Chakravarti) *Given an instance* $\pi = \{G, \mathbb{S}, z\}$ *of a DOP* Π *with min-sum objective function, a least cost solution* S^* *to* π*, and an element* $e \in G$*, the upper tolerance limit* β_e *is given by*

$$\beta_e = \begin{cases} \infty & \textit{if } e \notin S^* \\ z(S_e) - z(S^*) & \textit{if } e \in S^* \end{cases};$$

and the lower tolerance limit α_e *is given by*

$$\alpha_e = \begin{cases} z(S^e) - z(S^*) & \textit{if } e \notin S^* \\ \infty & \textit{if } e \in S^* \end{cases}.$$

We show below that this characterization follows from the properties of θ discussed in Subsection 2.1. Recall that θ is that value of c_e for which $z(S_e) = z(S^e)$, and in fact $\theta = F(S^e) - F(S_e)$. If the random element $e \in S^*$, then $c_e \in (-\infty, \theta]$. We have seen that in this case, S^* remains optimal regardless of the amount by which we decrease c_e but becomes suboptimal when $c_e > \theta$. If $e \notin S^*$, then $c_e \in [\theta, \infty)$. In this case, S^* remains optimal regardless of the amount by which we increase c_e and becomes suboptimal when $c_e < \theta$. This immediately leads to the characterization in Result 3.

DOPs with multiple random elements are studied in the literature on stability analysis (refer, for example, to Chakravarti & Wagelmans [1], Kravchenko *et al.* [4], Sotskov [8, 9], Sotskov *et al.* [6], Sotskov *et al.* [7]). Stability analysis aims to calculate regions in the Euclidean parameter space within which a given solution remains optimal. One of the common ways of expressing the stability region is the stability radius, which is defined as the maximal value of a variable ρ such that the cost of each random parameter can be changed by ρ without affecting the optimality of the solution at hand. In the remainder of this subsection we will consider a special case when two elements in a DOP instance are random and show how $\mathcal{R}_i$'s could be used to determine the stability radius in this case.

Let us consider a DOP instance π in which elements e_1 and e_2 are random. Suppose that we are given an least cost solution S^* to this instance when the costs of the random elements e_1 and e_2 are x_1 and x_2 respectively (denoted as the point $(x_1, x_2) \in \mathfrak{R}^2$). We define the sets $K_1 = \{1, 2\}$, $K_2 = \{1\}$, $K_3 = \{2\}$, and $K_4 = \emptyset$, and following (4), partition $\mathbb{S}$ into $\mathbb{S}_1, \ldots, \mathbb{S}_4$ and define $S_1, \ldots, S_4$ and $\mathcal{R}_1, \ldots, \mathcal{R}_4$ according to (5). (Notice that one of S_1 through S_4 will be indistinguishable from S^* by its response to changes in c_{e_1} and c_{e_2}.) It can be shown (refer to Appendix for formal proofs) that the regions $\mathcal{R}_1, \ldots, \mathcal{R}_4$ look either like Figure 2(a) or like Figure 2(b).

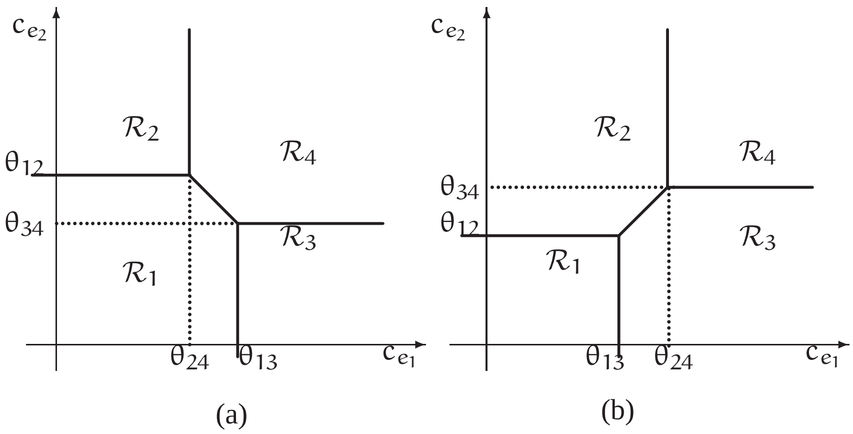

Figure 2: Shapes of sets $\mathcal{R}_1, \ldots, \mathcal{R}_4$

The actual shapes of $\mathcal{R}_1, \ldots, \mathcal{R}_4$ in $\Re^2$ can be ascertained by calculating the values of θ_{12} and θ_{34} (or θ_{13} and θ_{24}) in Figure 2. The θ's denote critical values of c_{e_1} and c_{e_2}. For instance, if c_{e_1} ($= x_1$) is fixed at low enough value so that $e_1 \in S^*$, then θ_{12} is the value of c_{e_2} such that $e_2 \in S^*$ iff $x_e \leq \theta_{12}$. Note that $\theta_{13} + \theta_{34} = \theta_{12} + \theta_{24}$. If $\theta_{12} > \theta_{34}$ (or $\theta_{13} > \theta_{24}$), then the shapes of $\mathcal{R}_1, \ldots, \mathcal{R}_4$ correspond to Figure 2(a), otherwise they correspond to Figure 2(b). To determine this, we calculate the least cost solutions at four points in $\Re^2$; $A = (-M, -M)$, $B = (-M, M)$, $C = (M, M)$, and $D = (M, -M)$ where M is large. If M is chosen sufficiently large, then $A \in \mathcal{R}_1$, $B \in \mathcal{R}_2$, $C \in \mathcal{R}_4$, and $D \in \mathcal{R}_3$. Let the optimal objective function value (in the least cost sense) of π at these four points be Z_A, Z_B, Z_C, and Z_D respectively. Then

$$\begin{aligned}
\theta_{12} &= M + (Z_B - Z_A), \\
\theta_{34} &= M + (Z_C - Z_D), \\
\theta_{23} &= -M + (Z_C - Z_B), \text{ and} \\
\theta_{14} &= -M + (Z_D - Z_A).
\end{aligned}$$

An inspection of S^* will tell us to which among $\mathcal{R}_1$ through $\mathcal{R}_4$ it belongs. We can then use these θ values to compute to stability radius of S^* as in Table 3.1.

Table 3.1: Stability Radius for $k = 2$

	$\theta_{12} > \theta_{34}$	$\theta_{12} \leq \theta_{34}$
$(x_1, x_2) \in \mathcal{R}_1$	$\rho = \min(\theta_{13} - x_1, \theta_{12} - x_2,$ $\theta_{12} + \theta_{24} - x_1 - x_2)$	$\rho = \min(\theta_{13} - x_1, \theta_{12} - x_2)$
$(x_1, x_2) \in \mathcal{R}_2$	$\rho = \min(\theta_{24} - x_1, x_2 - \theta_{12})$	$\rho = \min(\theta_{24} - x_1, x_2 - \theta_{12},$ $x_2 - x_1 - \theta_{12} + \theta_{13})$
$(x_1, x_2) \in \mathcal{R}_3$	$\rho = \min(x_1 - \theta_{13}, \theta_{34} - x_2)$	$\rho = \min(x_1 - \theta_{13}, \theta_{34} - x_2,$ $\theta_{12} - \theta_{13} - x_2 + x_1)$
$(x_1, x_2) \in \mathcal{R}_4$	$\rho = \min(x_1 - \theta_{24}, x_2 - \theta_{34},$ $x_1 + x_2 - \theta_{12} - \theta_{24})$	$\rho = \min(x_1 - \theta_{24}, x_2 - \theta_{34})$

3.3 DOPs with Min-Max Objective

Min-max (or bottleneck) DOPs represent a popular class of DOPs that can be used to model various practical situations. In these problems, the objective function of any solution is the cost of the largest cost element in the solution, i.e.

$$z(S) = \max_{e \in S}(c_e). \tag{15}$$

The optimization problem, as usual, is to find a solution with the minimum possible objective function value. DOPs with min-max objective functions occur frequently in the literature on logistics and machine scheduling.

In the analysis in this subsection, we will continue to use the same set of notations as in Section 2 with the following alterations and additions. Given $S \in \mathbb{S}$, we define

$$F(S) = \max_{e \in S \cap G_F} \{c_e\}; \quad V(S) = \max_{e \in S \cap G_R} \{c_e\} = \max_{i : e_i \in S \cap G_R} \{X_i\}.$$

From (15) and the definitions above, it is obvious that

$$z(S) = \max\left[F(S), V(S)\right].$$

Let us initially consider such a DOP with a single random element, say e. We can partition the set of solutions $\mathbb{S}$ into $\mathbb{S}_e$ and $\mathbb{S}^e$, as before, consisting of solutions that

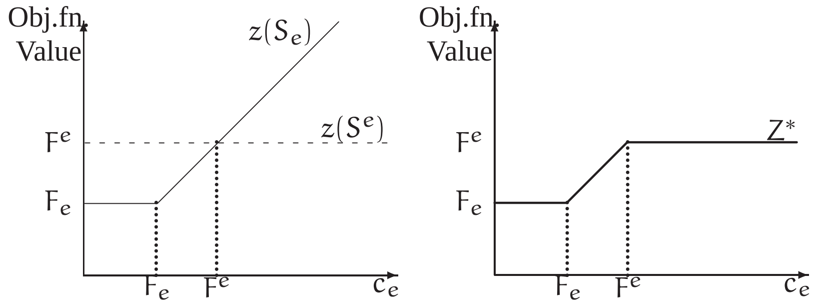

Figure 3: $z(S_e)$, $z(S^e)$, and Z^* as a function of c_e (min-max objective)

include e and those that do not. Let S^e denote a least cost solution in $\mathbb{S}^e$. Let S_e denote a solution in $\mathbb{S}_e$ such that

$$F(S_e) = \min_{S \in \mathbb{S}_e} F(S).$$

Then, if $c_e \leq F(S_e)$, we have,

$$z(S_e) = F(S_e) \leq F(S) \leq z(S) \quad \forall S \in \mathbb{S}_e;$$

and $c_e > F(S_e)$ implies that $z(S_e) = c_e$ while for any other $S \in \mathbb{S}_e$, $z(S) = \max(F(S), c_e) \geq c_e = z(S_e)$. Thus, S_e is a least cost solution in $\mathbb{S}_e$ irrespective of the cost of e. Hence, as before, we can restrict our search among S_e and S^e.

Let $F_e = F(S_e)$ and $F^e = z(S^e) = F(S^e)$. If $F_e \geq F^e$ the analysis is trivial and uninteresting, because then S^e is always optimal in the least risk sense. In the more interesting scenario of $F_e < F^e$, the optimal objective function value (in the least cost sense) looks like Figure 3.

The risks for the two candidates now turn out to be:

$$\begin{aligned}
R(S^e) &= \int_{-\infty}^{F^e} (F^e - z(S_e))dH(x) \\
&= \int_{-\infty}^{F_e} (F^e - F_e)dH(x) + \int_{F_e}^{F^e} (F^e - x)dH(x), \quad \text{and} && (16) \\
R(S_e) &= \int_{F^e}^{\infty} (x - F^e)dH(x). && (17)
\end{aligned}$$

So S^e is optimal in the least risk sense iff $R(S^e) \leq R(S_e)$, i.e., iff

$$\begin{aligned} & R(S_e) - R(S^e) \geq 0 \\ \iff \quad & \int_{-\infty}^{\infty} (x - F^e) dH(x) + \int_{-\infty}^{F_e} (F_e - x)) dH(x) \geq 0 \\ \iff \quad & F^e - \int_{-\infty}^{F_e} (F_e - x) dH(x) \leq \mu. \end{aligned} \tag{18}$$

Note that the characterization (18) depends on the knowledge of H in a way that is more involved than just knowledge of the mean. Indeed, it depends on the specific instance structure (namely $F(S_e)$). Thus, verifying the inequality in (18) is difficult, and this certainly limits its possible usage.

REMARK. Let $\delta = \int_{-\infty}^{F_e} (F_e - x) dH(x)$. Then $\delta \geq 0$, with equality holding iff

$$H(F_e) = 0 \iff P[c_e > F(S_e)] = 1. \tag{19}$$

(19) describes perhaps, the only case in which the solution reduces to a characterization similar to the min-sum case. Otherwise (for $\delta > 0$), whenever $\mu \in (F^e - \delta, F^e)$, we have a situation in contradiction to Theorem 2, as $z(S_e) < z(S^e)$ with $c_e = \mu$ and yet $R(S_e) > R(S^e)$.

In principle, it is not difficult to extend Result (18) for more than one random element. We can proceed as in Subsection 2.2 with the only major modification being that now, S_i needs to be a minimum *fixed* cost solution within $\mathbb{S}_i$, i.e.

$$F(S_i) = \min_{S \in \mathbb{S}_i} F(S).$$

We omit the details, because not only the characterization is messy, but it is also of very limited practical importance as is seen above even in the case with a single random element.

4. Summary and Directions for Future Research

In this paper, we considered the problem of solving a general class of discrete optimization problems with min-sum objective functions and having random elements.

The probability distribution of the costs of the random elements were assumed to be known. Defining the risk associated with feasible solutions as the expected value of their sub-optimality, we showed in Section 2, that if we solve the DOP after pegging the costs of the random elements to the mean values of the coresponding distribution functions, we obtain a solution that has the minimum risk. It is interesting to note that this result holds true irrespective of whether the random cost variables are independent or not. In Subsection 3.1 we discussed the course of action when the means of the random elements are not known and showed that it did not really matter provided we have 'enough' data. In Subsection 3.2, we drew comparison between our findings and results from post-optimality analysis. In Subsection 3.3 we found out that, if the objective function is min-max instead of min-sum, then the situation is much more complex.

In the current work, we have worked under the framework that the set of random elements (namely, G_R) remain the same throughout. In reality, it is quite possible that an element (say e) which is regarded as fixed, is in fact random. To avoid this, in our framework we can take $G_R = G$ with elements in G_S specified as having degenerate distribution. Now, it is easy to see, based on our results from the min-sum problem, that mistakes of this kind do not induce any penalty as long as the change from c_e to the mean of the probability distribution of cost of that random element does not amount to a change in the partition of the Euclidean space to which the mean vector belongs. Unfortunately, it is impossible to measure the impact of such changes in quantitative terms as it is likely to be problem specific. In future, we plan to investigate if it is possible to derive characterization of elements which are more susceptible in that regard.

Another interesting variation of the current setup is where the set of feasible solution $\mathbb{S}$ may itself be random. As an illustration, consider the 0/1 Knapsack Problem in Example 2 with the variation that some of the weights w_j and/or the capacity B is now random. The treatment edscribed in the current work clearly does not permit such a scenario, but the problem is of practical significance and we plan to study this in future.

A third interesting direction of research is towards discrete optimization problems with min-max objectives and having random cost elements. The current work has shown that such problems become intractible with the methods adopted here. However, other methods, throwing fresh light on the behaviour of these problems cannot be ruled out.

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Appendix

In this appendix, we formally prove that the shapes of $\mathcal{R}_1, \ldots, \mathcal{R}_4$ correspond to either Figure 2(a) or 2(b). Recall that the $\mathcal{R}_i$'s have been defined in Subsection 2.2 through specific choices of S_i's within $\mathbb{S}_i$'s. But, while the S_i, (the least cost solution within $\mathbb{S}_i$) need not be unique, it is easy to see that the fixed cost components of the candidate S_i's (for fixed i) are the same, and hence $\mathcal{R}_i$'s remain unaffected by their choices. We will use the term $z_{(x_1,x_2)}(S)$ to denote the objective function value of a solution S at the point $(x_1, x_2) \in \mathfrak{R}^2$ (i.e. when the two random costs are $X_1 = x_1$ and $X_2 = x_2$). We will also use S^* to denote a least cost solution at (x_1, x_2).

Notice that the interiors of $\mathcal{R}_i$'s are disjoint. We show this for the intersection between $\mathcal{R}_1$ and $\mathcal{R}_2$, but the argument is similar for any such pair. If the two interiors intersect, then there would exist a $\delta > 0$ and a rectangle $[a, a+\delta] \times [b, b+\delta]$ contained in $\mathcal{R}_1 \cap \mathcal{R}_2$. This implies that, there exists $S_1 \in \mathbb{S}_1$ and $S_2 \in \mathbb{S}_2$ s.t.

$$z_{x,y}(S_1) = z_{x,y}(S_2), \quad \forall (x, y) \in [a, a+\delta] \times [b, b+\delta];$$

but $z_{x,y}(S_1) = F(S_1) + x + y$ and $z_{x,y}(S_2) = F(S_2) + x$, and hence, we must have

$$F(S_1) - F(S_2) = y, \quad \forall y \in [b, b+\delta],$$

which leads to a contradiction.

Consequently, the overlap between any two $\mathcal{R}_i$'s can only be a line and would be referred to as the boundary between the two regions.

Lemma 1 *Let $\delta > 0$. Then*

(a) *if $(x_1, x_2) \in \mathcal{R}_1$ then $(x_1 - \delta, x_2) \in \mathcal{R}_1$,*
(b) *if $(x_1, x_2) \in \mathcal{R}_1$ then $(x_1, x_2 - \delta) \in \mathcal{R}_1$,*
(c) *if $(x_1, x_2) \in \mathcal{R}_2$ then $(x_1 - \delta, x_2) \in \mathcal{R}_2$,*
(d) *if $(x_1, x_2) \in \mathcal{R}_2$ then $(x_1, x_2 + \delta) \in \mathcal{R}_2$,*
(e) *if $(x_1, x_2) \in \mathcal{R}_3$ then $(x_1 + \delta, x_2) \in \mathcal{R}_3$,*
(f) *if $(x_1, x_2) \in \mathcal{R}_3$ then $(x_1, x_2 - \delta) \in \mathcal{R}_3$,*
(g) *if $(x_1, x_2) \in \mathcal{R}_4$ then $(x_1 + \delta, x_2) \in \mathcal{R}_4$, and*
(h) *if $(x_1, x_2) \in \mathcal{R}_4$ then $(x_1, x_2 + \delta) \in \mathcal{R}_4$.*

PROOF. All the statements can be proved by contradiction. We only prove part (a), the remaining statements can be proved using similar arguments.

Suppose the statement to be false. Then there exists a solution $S \notin \mathcal{R}_1$ such that $z_{(x_1-\delta,x_2)}(S) < z_{(x_1-\delta,x_2)}(S^*) = z_{(x_1,x_2)}(S^*) - \delta$. Since $S \notin \mathcal{R}_1$, either $e_1 \notin S$ or $e_2 \notin S$. If $e_1 \notin S$, then $z_{(x_1,x_2)}(S) = z_{(x_1-\delta,x_2)}(S) < z_{(x_1,x_2)}(S^*)$ which is a contradiction. If $e_2 \notin S$, then $z_{(x_1,x_2)}(S) \leq z_{(x_1-\delta,x_2)}(S) + \delta < z_{(x_1,x_2)}(S^*)$. This contradicts S^* being optimal at (x_1, x_2). Hence our postulated S cannot exist, i.e. $(x_1-\delta, x_2) \in \mathcal{R}_1$. $\square$

Lemma 1 shows that the $\mathcal{R}_i$'s are convex and helps to determine the relative positioning of $\mathcal{R}_1$ through $\mathcal{R}_4$ with respect in $\mathfrak{R}^2$. In particular, it shows that $\mathcal{R}_3$ and $\mathcal{R}_4$ lies to the right of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, and $\mathcal{R}_2$ and $\mathcal{R}_4$ lies above $\mathcal{R}_1$ and $\mathcal{R}_3$, respectively.

Statements (a) & (c) together show that the boundary between $\mathcal{R}_1$ & $\mathcal{R}_2$ is a straight line with slope 0, statements (b) & (f) show that the boundary between $\mathcal{R}_1$ & $\mathcal{R}_3$ is a straight line with slope ∞, statements (e) & (g) together show that the boundary between $\mathcal{R}_3$ & $\mathcal{R}_4$ is a straight line with slope 0, and statements (d) & (h) together show that the boundary between $\mathcal{R}_2$ & $\mathcal{R}_4$ is a straight line with slope ∞.

Geometrically, it is obvious that the boundaries between $\mathcal{R}_1$ & $\mathcal{R}_4$ and between $\mathcal{R}_2$ & $\mathcal{R}_3$ cannot simultaneously be strictly positive. In fact the first of these two boundaries have positive length iff $\theta_{12} \geq \theta_{34}$ (and $\theta_{13} \geq \theta_{24}$) while the second boundary has a positive length iff the inequalities change direction. The next lemma shows that the boundary between $\mathcal{R}_1$ & $\mathcal{R}_4$ has slope -1, and that between $\mathcal{R}_2$ & $\mathcal{R}_3$ has slope 1.

Lemma 2

(a) If the boundary between $\mathcal{R}_1$ and $\mathcal{R}_4$ has a positive length, then it is a straight line with slope -1.

(b) If the boundary between $\mathcal{R}_2$ and $\mathcal{R}_3$ has a positive length, then it is a straight line with slope 1.

PROOF. We prove the part (a), with part (b) following from a similar string of arguments. Since the boundary belongs to $\mathcal{R}_4$, the least cost solutions at any point (say, (x_1, x_2)) in the boundary have the same objective value (say, c). But each of these points also belong to $\mathcal{R}_1$. Thus,

$$z_{(x_1,x_2)}(S) = F(S) + x_1 + x_2,$$

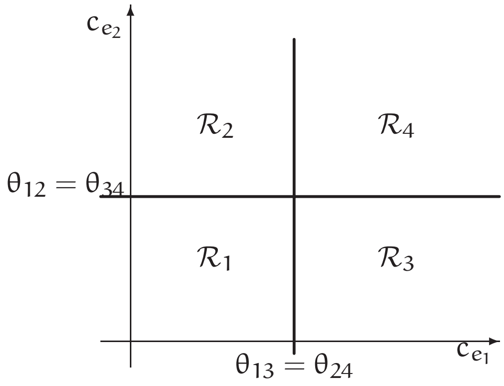

Figure 4: A special case of the shapes of $\mathcal{R}_1, \ldots, \mathcal{R}_4$

and hence, boundary must be part of the line having equation

$$x_1 + x_2 = c - F(S).$$

□

From the characterizations above, we see that there can be one of two cases depicted in Figure 2. (As a special case, we can also have a situation (see Figure 4) where both the boundaries between $\mathcal{R}_1$ & $\mathcal{R}_4$ and between $\mathcal{R}_2$ & $\mathcal{R}_3$ have zero length.)